Linear response of galactic halos to adiabatic gravitational perturbations

Chigurupati Murali[1] and Scott Tremaine[1,2]

[1]Canadian Institute for Theoretical Astrophysics, McLennan Labs, University of Toronto, 60 St. George St., Toronto M5S 3H8, Canada

[2]Canadian Institute for Advanced Research, Program in Cosmology and Gravity

ABSTRACT

We determine the response of a self-similar isothermal stellar system to small adiabatic gravitational perturbations. For odd spherical harmonics, the response is identical to the response of the analogous isothermal fluid system. For even spherical harmonics, the response can be regarded as an infinite series of wavetrains in $\log r$, implying alternating compression and rarefaction in equal logarithmic radius intervals. Partly because of the oscillatory nature of the solutions, tidal fields from external sources are not strongly amplified by an intervening isothermal stellar system, except at radii $\lesssim 10^{-3.5}$ times the satellite radius; at some radii the stellar system can even screen the external tidal field in a manner analogous to Debye screening. As Weinberg has pointed out, individual resonances in a stellar system can strongly amplify external tidal fields over a limited radial range, but we cannot address this possibility because we examine only adiabatic perturbations. We also discuss the application of our method to the halo response caused by the slow growth of an embedded thin disk.

Subject headings: stellar dynamics – galaxies: individual (Milky Way) – galaxies: haloes – galaxies: kinematics and dynamics

1. Introduction

In models of galaxy formation based on hierarchical clustering, the gravitational field becomes less and less smooth at larger and larger distances from the galaxy center. The halos of isolated galaxies contain satellite and companion galaxies and merging dark matter substructure, and galaxies in groups and clusters are subject to slowly varying tidal fields from distant group members as well as rapidly changing forces from close encounters. Despite

this noisy environment, the visible inner regions of most galaxies are relatively smooth and regular.

One expects that gravitational noise from the outer halo and beyond will not severely disturb the inner galaxy because the dominant (quadrupole) tidal potential at radius r from a source of mass m_p at radius $r_p \gg r$ varies as $Gm_p r^2/r_p^3$; in other words the ratio of the tidal force to the force from the body of the galaxy itself is of order $(m_p/m)(r/r_p)^3$ where m is the mass of the galaxy within r. The cubic dependence of the force ratio on the radius ratio implies that tidal forces from distant satellites are generally unable to strongly perturb the inner galaxy ($r \ll r_p$).

However, this assessment neglects an important effect: in a realistic galaxy model the tidal forces do not propagate through a vacuum—rather, they propagate through the density field of the dark-matter halo. The halo provides an intervening medium that modifies the external tidal field in a manner analogous to the polarization of a dielectric medium in electrostatics (Jackson 1975). Another closely related analogue is the dimensionless Love number in geophysics (e.g. Jeffreys 1970), which measures the ratio of the direct tidal potential from the Moon to the augmented tidal potential that includes the gravitational potential arising from the deformation of the Earth in response to the lunar tide.

The influence of the halo response on tidal fields was first pointed out by Lynden-Bell (1985), who computed the Love number for a spherical fluid halo with power-law density distribution, $\rho_0(r) \propto r^{-\alpha}$ (see also Nelson & Tremaine 1995); we shall find below, however, that the response of fluid and stellar systems with the same density distribution can be quite different. Weinberg (1995, 1997) found that the response of the Galactic halo strongly enhances the direct tidal field from the Magellanic Clouds and suggested that the resulting disk distortion could account for the location, position angle, and sign of the HI warp in the outer Galaxy. In many respects Weinberg's calculations are much more sophisticated than ours, as we shall focus on a simplified case that provides insight into the phenomenon rather than accurate numbers for a realistic system.

In the present paper, we examine the response of a spherical, scale-free isothermal stellar system to an adiabatically applied external gravitational perturbation. This problem only approximates reality in that it neglects the time-dependence that accompanies most tidal fields (e.g. those due to orbiting satellites). The reward is an analytically tractable problem that permits us to understand the physics of the halo response—and the linear response of stellar systems in general—in a more detailed manner than has hitherto been possible. Moreover, we are most interested in the response of the galaxy at small radii, where the dynamical time is short and the approximation of a static tide is not unreasonable.

Adiabatically applied gravitational perturbations are also relevant to other astrophysical phenomena, including the slow growth of a central black hole in a galaxy (e.g. Young 1980, Quinlan et al. 1995) and the slow growth of a galaxy disk in a halo or spheroid (Binney & May 1986; Dubinski 1994). Thus we shall also apply the tools we have developed to these problems.

The plan of the paper is as follows. In §2, we develop the solution to the linearized Boltzmann-Poisson equation that describes the response of a stellar system to an adiabatically applied perturbation. The relationship between adiabaticity and reversibility is discussed and a comparison with analogous fluid systems is also made. In §3, we derive the adiabatic Green's function for the response of the isothermal sphere. This machinery is applied in §4 and the results are discussed in §5.

2. Linearized Adiabatic Response

We examine the response of a stationary stellar system with unperturbed distribution function (hereafter DF) $F(\mathbf{r},\mathbf{v})$ and gravitational potential $\Phi_0(\mathbf{r})$ to a weak external potential $\Phi^e(\mathbf{r},t)$. The evolution of the perturbed DF $f(\mathbf{r},\mathbf{v},t)$ is described by the linearized collisionless Boltzmann equation (e.g. Kalnajs 1971; Weinberg 1989),

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} - \boldsymbol{\nabla}\Phi_0\cdot\frac{\partial f}{\partial \mathbf{v}} = \boldsymbol{\nabla}\Phi_1\cdot\frac{\partial F}{\partial \mathbf{v}}; \tag{1}$$

the first equality defines the convective or Lagrangian derivative in phase space. The perturbing potential $\Phi_1(\mathbf{r},t) = \Phi^e(\mathbf{r},t) + \Phi^s(\mathbf{r},t)$, where Φ^s is the response potential, given by Poisson's equation

$$\nabla^2\Phi^s = 4\pi G\int f d\mathbf{v}. \tag{2}$$

We shall restrict ourselves to the case where the DF of the stationary stellar system depends only on energy, $F(\mathbf{r},\mathbf{v}) = F(E)$, where $E = \frac{1}{2}\mathbf{v}^2 + \Phi_0(\mathbf{r})$ is the energy per unit mass. Then equation (1) becomes

$$\frac{df}{dt} = \mathbf{v}\cdot\boldsymbol{\nabla}\Phi_1\frac{dF}{dE} = F_E\left(\frac{d\Phi_1}{dt} - \frac{\partial\Phi_1}{\partial t}\right), \tag{3}$$

where $F_E \equiv dF/dE$. Note that $-\mathbf{v}\cdot\boldsymbol{\nabla}\Phi_1$ is the power per unit mass delivered by the perturbing potential.

In this paper we shall focus on the case where the external perturbation grows slowly from zero in the distant past. We assume that the potential $\Phi_0(\mathbf{r})$ admits action-angle

variables $(\mathbf{I}, \mathbf{w})$, so that the unperturbed stellar orbits have the form $\mathbf{I}$ =constant, $\mathbf{w} = \boldsymbol{\omega}(\mathbf{I})t$+constant. We then expand the perturbing potential in a Fourier series of the form $\Phi_1 = \sum_{\mathbf{k}} \Phi_{\mathbf{k}}(\mathbf{I}, t) \exp(i\mathbf{k} \cdot \mathbf{w})$, where $\mathbf{k}$ is an integer triplet. We assume that $\Phi_{\mathbf{k}}(\mathbf{I}, t) = \Phi_{\mathbf{k}}(\mathbf{I}) \exp(\epsilon t)$ where ϵ is small and positive. Then we can integrate (3) to obtain

$$f = F_E \sum_{\mathbf{k}} \Phi_{\mathbf{k}}(\mathbf{I}, t) \left[\frac{\mathbf{k} \cdot \boldsymbol{\omega}}{\mathbf{k} \cdot \boldsymbol{\omega} - i\epsilon}\right] \exp(i\mathbf{k} \cdot \mathbf{w}). \tag{4}$$

In the limit $\epsilon \to 0$ the square bracket approaches unity, unless $\mathbf{k} \cdot \boldsymbol{\omega} = 0$, in which case it is zero. The condition $\mathbf{k} \cdot \boldsymbol{\omega} = 0$ can be satisfied in three different ways:

1. At isolated or local resonances in phase space, where $\mathbf{k} \cdot \boldsymbol{\omega}(\mathbf{I}) = 0$ for a particular set of actions $\mathbf{I}$ (accidental degeneracies). We shall ignore such resonances since they are unlikely to dominate the overall response of the stellar system.

2. When $\mathbf{k} = \mathbf{0}$.

3. When some symmetry of the stellar system dictates that there are non-zero values of $\mathbf{k}$ such that $\mathbf{k} \cdot \boldsymbol{\omega}(\mathbf{I}) = 0$ for all values of the actions $\mathbf{I}$ (global resonance). For example, with a suitable choice of actions, in spherical potentials $\omega_3 = 0$ (the orbital plane does not precess) so that $\mathbf{k} \cdot \boldsymbol{\omega} = 0$ whenever $k_1 = k_2 = 0$; and for Kepler potentials $\omega_1 = \omega_2$ so that $\mathbf{k} \cdot \boldsymbol{\omega} = 0$ whenever $k_1 = -k_2$.

For a given stellar system, we shall denote by N the set of integer triples $\mathbf{k}$ such that $\mathbf{k} \cdot \boldsymbol{\omega}(\mathbf{I}) = 0$ for all values of the actions $\mathbf{I}$ (i.e. all $\mathbf{k}$ satisfying conditions 2 or 3 above). Then in the adiabatic limit $\epsilon \to 0$ equation (4) becomes (e.g. Lynden-Bell 1969)

$$f = F_E \sum_{\mathbf{k} \notin N} \Phi_{\mathbf{k}}(\mathbf{I}, t) \exp(i\mathbf{k} \cdot \mathbf{w}) = F_E[\Phi_1 - \langle \Phi_1 \rangle], \tag{5}$$

where $\langle \Phi_1 \rangle = \sum_{\mathbf{k} \in N} \Phi_{\mathbf{k}}(\mathbf{I}, t) \exp(i\mathbf{k} \cdot \mathbf{w})$. $\langle \Phi_1 \rangle$ can be regarded as a time average of the perturbing potential, over times long compared to the orbital period but short compared to ϵ^{-1}. This is not the same as an average over orbital phase, because the time average of $\exp(i\mathbf{k} \cdot \mathbf{w}) \neq 0$ if $\mathbf{k} \in N$ but the phase average of $\exp(i\mathbf{k} \cdot \mathbf{w}) \neq 0$ only if $\mathbf{k} = \mathbf{0}$.

Note that the solution (5) is only one of many possible static solutions to the linearized collisionless Boltzmann equation if the set N is not empty. If we write the perturbed DF in action-angle variables, $f = \sum_{\mathbf{k}} f_{\mathbf{k}}(\mathbf{I}) \exp(i\mathbf{k} \cdot \mathbf{w})$, then the linearized collisionless Boltzmann equation reads

$$\sum_{\mathbf{k}} \exp(i\mathbf{k} \cdot \mathbf{w})(\mathbf{k} \cdot \boldsymbol{\omega})(f_{\mathbf{k}} - F_E \Phi_{\mathbf{k}}) = 0, \tag{6}$$

which has the general solution

$$f = F_E \Phi_1 + \sum_{\mathbf{k} \in N} p_{\mathbf{k}}(\mathbf{I}) \exp(i\mathbf{k} \cdot \mathbf{w}) \tag{7}$$

where $p_{\mathbf{k}}(\mathbf{I})$ is an arbitrary function; equation (5) corresponds to the particular case $p_{\mathbf{k}} = -F_E \Phi_{\mathbf{k}}$.

The absence of the term $\mathbf{k} = \mathbf{0}$ in equation (5) has a simple physical interpretation in terms of entropy. The entropy of the unperturbed stellar system is

$$S_i = \int d\mathbf{r} d\mathbf{v} F \ln F = (2\pi)^3 \int d\mathbf{I} F \ln F, \tag{8}$$

because $F(E)$ is a function of the actions only and $d\mathbf{r}d\mathbf{v} = d\mathbf{I}d\mathbf{w}$. The final entropy is

$$S_f = \int d\mathbf{I}d\mathbf{w}(F+f)\ln(F+f) = S_i + \int d\mathbf{I}d\mathbf{w}(1+\ln F)f + \mathrm{O}(f^2). \tag{9}$$

Using equation (5), we find the change in entropy

$$\Delta S \equiv S_f - S_i = \int d\mathbf{I}d\mathbf{w}(1+\ln F)F_E \sum_{\mathbf{k} \notin N} \Phi_{\mathbf{k}} \exp(i\mathbf{k} \cdot \mathbf{w}), \tag{10}$$

which vanishes upon integrating over $\mathbf{w}$, because $\mathbf{0} \in N$.

We can also compare the solution (5) to the response of a barotropic fluid system to a weak, slowly growing external potential. In an equilibrium barotropic fluid, the pressure P, density ρ and potential Φ form a one-parameter family under an isentropic change; thus $P = P(\rho) = P(\Phi)$. The equation of hydrostatic equilibrium states that $dP = -\rho d\Phi$ so

$$\rho = -\frac{dP}{d\Phi}. \tag{11}$$

For a small change in potential Φ_1, we can determine the perturbed density by a Taylor expansion of this equation. Defining $\rho = \rho_0 + \rho_1$, we find that

$$\rho_1 = -\left(\frac{d^2P}{d\Phi^2}\right)_0 \Phi_1 = \left(\frac{d\rho}{d\Phi}\right)_0 \Phi_1. \tag{12}$$

For comparison to stellar systems, take equation (7) with $p_{\mathbf{k}} = 0$, and integrate over velocity:

$$\rho_1 = \int f d\mathbf{v} = \Phi_1 \int F_E d\mathbf{v}. \tag{13}$$

Since the unperturbed density $\rho_0 = \int F(\frac{1}{2}v^2 + \Phi_0)d\mathbf{v}$, we have $(d\rho/d\Phi)_0 = \int F_E d\mathbf{v}$, so that

$$\rho_1 = \left(\frac{d\rho}{d\Phi}\right)_0 \Phi_1. \tag{14}$$

The result (14) is equivalent to (12). In other words the response of a barotropic fluid system to a slowly growing potential is the same as the static solution

$$f_1 = F_E \Phi_1 \tag{15}$$

of the stellar system with the same density distribution, but is *not* the same as the response of this stellar system to the same slowly growing potential.

In the following sections we shall examine the adiabatic response of both stellar (eq. 5) and fluid (eq. 15) systems; the fluid is less realistic but captures much of the relevant physics with less algebraic complexity.

3. Adiabatic response of an isothermal sphere

We now investigate the solutions to equations (2) and (5) in the case where the stationary stellar system is a singular isothermal sphere (Chandrasekhar 1939, Binney & Tremaine 1987), in which the DF $F(E)$, the density $\rho_0(r) = \int F d\mathbf{v}$, and the potential $\Phi_0(r)$ have the form

$$F(E) = K\exp(-\beta E), \qquad \rho_0(r) = \frac{\rho_a r_a^2}{r^2}, \qquad \Phi_0(r) = \frac{2}{\beta}\log(r/r_a), \tag{16}$$

where $2\pi G\rho_a r_a^2\beta = 1$, $K = (2\pi)^{-3/2}\rho_a\beta^{3/2}$. Equations (2) and (5) can be written as

$$\nabla^2\Phi^s + \frac{2}{r^2}(\Phi^s + \Phi^e) = -4\pi G\int d\mathbf{v} F_E \langle \Phi^s + \Phi^e\rangle_{\mathbf{r},\mathbf{v}}, \tag{17}$$

where $\langle\cdot\rangle_{\mathbf{r},\mathbf{v}}$ denotes an average—in the sense of equation (5)—over the orbit passing through $(\mathbf{r},\mathbf{v})$. This constitutes an integro-differential equation for the response potential Φ^s. In the case of a fluid the right side of equation (17) would be zero.

Write

$$\Phi(\mathbf{r}) = \sum_{\ell m} \Phi_{\ell m}(r) Y_{\ell m}(\mathbf{\Omega}). \tag{18}$$

Then after multiplying (17) by $Y^*_{\ell m}(\mathbf{\Omega})$ and integrating over $d\mathbf{\Omega}$ we get

$$\begin{aligned} &\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\Phi^s_{\ell m} + \frac{1}{r^2}[2-\ell(\ell+1)]\Phi^s_{\ell m} + \frac{2}{r^2}\Phi^e_{\ell m} \\ &\quad = \;\; -4\pi G\int d\mathbf{\Omega} Y^*_{\ell m}(\mathbf{\Omega})\int d\mathbf{v} F_E \left\langle \sum_{\ell' m'} \left(\Phi^s_{\ell' m'} + \Phi^e_{\ell' m'}\right) Y_{\ell' m'} \right\rangle_{\mathbf{r},\mathbf{v}}. \end{aligned} \tag{19}$$

We now multiply (19) by $r^{\alpha+1}$ and integrate over r. We assume that $\Phi^s_{\ell m} \sim r^{-a_-}$ as $r \to 0$ and $\sim r^{-a_+}$ as $r \to \infty$, where $a_+ \le \ell + 1$, $a_- \ge -\ell$ since otherwise the density distribution

that gives rise to $\Phi^s_{\ell m}$ is unphysical. We restrict α to the strip of the complex plane defined by

$$-\ell \leq a_- < Re(\alpha) < a_+ \leq \ell + 1, \tag{20}$$

so that boundary terms arising from integration by parts vanish.

The Mellin transform of a function $y(r)$ is written $\widetilde{y}(\alpha)$, where

$$\widetilde{y}(\alpha) = \int_0^\infty r^{\alpha-1} y(r) dr, \qquad y(r) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \widetilde{y}(\alpha) r^{-\alpha} d\alpha. \tag{21}$$

Thus

$$\begin{aligned} \alpha(\alpha-1)\widetilde{\Phi}^s_{\ell m}(\alpha) + [2-\ell(\ell+1)]\widetilde{\Phi}^s_{\ell m}(\alpha) + 2\widetilde{\Phi}^e_{\ell m}(\alpha) \\ = \; -4\pi G \int d\mathbf{r} d\mathbf{v} r^{\alpha-1} Y^*_{\ell m}(\mathbf{\Omega}) F_E \left\langle \sum_{\ell' m'} (\Phi^s_{\ell' m'} + \Phi^e_{\ell' m'}) Y_{\ell' m'} \right\rangle_{\mathbf{r},\mathbf{v}} . \end{aligned} \tag{22}$$

We now convert to action-angle variables $(\mathbf{I}, \mathbf{w})$. Our conventions follow Tremaine & Weinberg (1984): I_2 is the total angular momentum, I_3 is the z-component of the angular momentum, and I_1 is the radial action,

$$I_1 = \frac{1}{\pi}\int_{r_p}^{r_a} v_r dr, \tag{23}$$

where $v_r = [2E - 2\Phi_0(r) - I_2^2/r^2]^{1/2}$ is the radial velocity and r_p and r_a are the pericenter and apocenter distances, at which $v_r = 0$. The only angle with a direct geometrical interpretation is w_3, which is the azimuth at which the orbit crosses the equatorial plane upward (the ascending node). Motion in the unperturbed Hamiltonian $H_0(\mathbf{I}, \mathbf{w}) = \frac{1}{2}v^2 + \Phi_0(r)$ is given by $\mathbf{I}$ =constant, $\mathbf{w} = \boldsymbol{\omega} t + \mathbf{w}_0$ where $\boldsymbol{\omega} = \partial H_0/\partial \mathbf{I}$. Note that $\omega_3 = 0$, since the orbital plane does not precess in a spherical potential, so that $\mathbf{k}\cdot\boldsymbol{\omega} = 0$ if $k_1 = k_2 = 0$. Note also that $d\mathbf{r}d\mathbf{v} = d\mathbf{I}d\mathbf{w}$.

Expanding the potentials in action-angle variables, we have

$$\begin{aligned} [\Phi^s_{\ell' m'}(r') + \Phi^e_{\ell' m'}(r')] Y_{\ell' m'}(\mathbf{\Omega}') &= \sum_{l'_1 l'_2 l'_3} \delta_{l'_3 m'} V_{\ell' l'_2 l'_3}(\beta') W^{l'_1}_{\ell' l'_2 l'_3}(\mathbf{I}') \exp\left(i \sum_{k=1}^3 l'_k w'_k\right), \\ r^{\alpha^*-1} Y_{\ell m}(\mathbf{\Omega}) &= \sum_{l_1 l_2 l_3} \delta_{l_3 m} V_{\ell l_2 l_3}(\beta) U^{l_1}_{\ell l_2}(\mathbf{I}) \exp\left(i \sum_{k=1}^3 l_k w_k\right). \end{aligned} \tag{24}$$

Here $\cos\beta = I_3/I_2$, $V_{\ell l_2 l_3}(\beta)$ is defined in terms of rotation matrices by Tremaine & Weinberg (1984), and

$$\begin{aligned} W^{l_1}_{\ell l_2 m}(\mathbf{I}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} dw_1 \exp(-il_1 w_1)[\Phi^s_{\ell m}(r) + \Phi^e_{\ell m}(r)] \exp[-il_2\chi(\mathbf{I}, w_1)], \\ U^{l_1}_{\ell l_2}(\mathbf{I}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} dw_1 \exp(-il_1 w_1) r^{\alpha^*-1} \exp[-il_2\chi(\mathbf{I}, w_1)]. \end{aligned} \tag{25}$$

In this equation

$$\chi(\mathbf{I}, w_1) = \int_{r_p(\mathbf{I})}^{r(\mathbf{I},w_1)} \frac{dr}{v_r}(\omega_2 - I_2/r^2). \tag{26}$$

We now find the average $\langle\cdot\rangle_{\mathbf{r},\mathbf{v}}$ of the first of equations (24). This is obtained by replacing $\mathbf{I}'$ by $\mathbf{I}$, β' by β, w_k' by w_k and restricting the summation to $(l_1', l_2', l_3') \in N$. Now substitute this result into equation (22), replacing $d\mathbf{r}d\mathbf{v}$ by $d\mathbf{I}d\mathbf{w}$ and $r^{\alpha-1}Y_{\ell m}(\mathbf{\Omega})$ by the complex conjugate of the second of equations (24). Only terms with $l_1 = l_1'$, $l_2 = l_2'$, and $l_3 = l_3' = m = m'$ survive the integration over $\mathbf{w}$. Moreover, in the singular isothermal sphere the set of resonant triplets N is given by $l_1 = l_2 = 0$, since the only global resonance is due to spherical symmetry. Thus, the right side of (22) becomes

$$\begin{aligned} &-4\pi G \int d\mathbf{r}d\mathbf{v} r^{\alpha-1} Y_{\ell m}^*(\mathbf{\Omega}) F_E \left\langle \sum_{\ell' m'} (\Phi_{\ell' m'}^s + \Phi_{\ell' m'}^e) Y_{\ell' m'} \right\rangle_{\mathbf{r},\mathbf{v}} \\ &\quad = \quad -2^5\pi^4 G \sum_{\ell'} \int dI_1 I_2 dI_2 d\cos\beta V_{\ell 0 m}^*(\beta) U_{\ell 0}^{0*}(\mathbf{I}) F_E V_{\ell' 0 m}(\beta) W_{\ell' 0 m}^0(\mathbf{I}). \end{aligned} \tag{27}$$

We next use the orthogonality relation (Edmonds 1960)

$$\int d\cos\beta V_{\ell l_2 m}^*(\beta) V_{\ell' l_2 m}(\beta) = \frac{2}{2\ell+1} \left| Y_{\ell l_2}(\tfrac{1}{2}\pi, 0) \right|^2 \delta_{\ell\ell'} \equiv C_{\ell l_2} \delta_{\ell\ell'}; \tag{28}$$

in particular

$$\begin{aligned} C_{\ell 0} &= \frac{1}{2\pi^2} \left[\frac{\Gamma(\frac{1}{2}\ell + \frac{1}{2})}{\Gamma(\frac{1}{2}\ell + 1)} \right]^2, \qquad \ell \text{ even}, \\ &= 0, \qquad \ell \text{ odd}. \end{aligned} \tag{29}$$

Thus equation (22) becomes

$$\begin{aligned} &\alpha(\alpha-1)\widetilde{\Phi}_{\ell m}^s(\alpha) + [2 - \ell(\ell+1)]\widetilde{\Phi}_{\ell m}^s(\alpha) + 2\widetilde{\Phi}_{\ell m}^e(\alpha) \\ &= -2^5\pi^4 C_{\ell 0} G \int dI_1 I_2 dI_2 F_E U_{\ell 0}^{0*}(\mathbf{I}) W_{\ell 0 m}^0(\mathbf{I}) \\ &= -8\pi^2 C_{\ell 0} G \int dI_1 I_2 dI_2 F_E \int dw_1 r^{\alpha-1}(\mathbf{I}, w_1) \int dw_1' \left(\Phi_{\ell m}^s + \Phi_{\ell m}^e\right) [r(\mathbf{I}, w_1')] \\ &= -8\pi^2 C_{\ell 0} G \int \frac{R dR dE I_c^2(E) F_E}{\omega_1(E,R)} \int dw_1 r^{\alpha-1}(\mathbf{I}, w_1) \int dw_1' \left(\Phi_{\ell m}^s + \Phi_{\ell m}^e\right) [r(\mathbf{I}, w_1')]; \end{aligned} \tag{30}$$

in the last line we have changed the integration variables from I_1, I_2 to the energy E and the dimensionless angular momentum $R \equiv I_2/I_c(E)$, where $I_c(E)$ is the angular momentum of a circular orbit of energy E.

Since the potential is scale-free the radius can be written in the form $r(\mathbf{I}, w_1) = r_c(E) \times x(R, w_1)$, where x is a dimensionless function that can be determined by numerical integration of the orbits. Changing the integration variable from E to $r' = r_c(E)x(w_1', R)$, the right side of (30) becomes

$$-8\pi^2 C_{\ell 0} G \int RdR \int dw_1 x^{\alpha-1}(R, w_1) \\ \times \int dw_1' x^{-\alpha}(R, w_1') \int dr' r'^{\alpha-1} \left[\Phi^s_{\ell m}(r') + \Phi^e_{\ell m}(r')\right] \frac{dE}{dr_c} \frac{I_c^2(E) F_E}{\omega_1(E, R)}, \tag{31}$$

For the singular isothermal sphere

$$\frac{dE}{dr_c} \frac{I_c^2(E) F_E}{\omega_1(E, R)} = -\frac{1}{2^{1/2}\pi^{5/2} e G g(R)}, \qquad \text{where} \qquad g(R) \equiv \omega_1(E, R) r_c(E) \beta^{1/2} \tag{32}$$

is a function only of R. Thus

$$\begin{aligned} &\alpha(\alpha-1)\widetilde{\Phi}^s_{\ell m}(\alpha) + [2 - \ell(\ell+1)]\widetilde{\Phi}^s_{\ell m}(\alpha) + 2\widetilde{\Phi}^e_{\ell m}(\alpha) \\ &\quad = \frac{2^{5/2} C_{\ell 0}}{\pi^{1/2} e} \int \frac{RdR}{g(R)} \int dw_1 x^{\alpha-1}(R, w_1) \int dw_1' x^{-\alpha}(R, w_1') \left[\widetilde{\Phi}^s_{\ell m}(\alpha) + \widetilde{\Phi}^e_{\ell m}(\alpha)\right] \\ &\quad \equiv C_{\ell 0} H(\alpha) \left[\widetilde{\Phi}^s_{\ell m}(\alpha) + \widetilde{\Phi}^e_{\ell m}(\alpha)\right], \end{aligned} \tag{33}$$

where

$$H(\alpha) = \frac{2^{5/2}}{\pi^{1/2} e} \int \frac{RdR}{g(R)} \int dw_1 x^{\alpha-1}(R, w_1) \int dw_1' x^{-\alpha}(R, w_1'). \tag{34}$$

Note that

$$H^*(\alpha) = H(\alpha^*), \qquad H(1-\alpha) = H(\alpha). \tag{35}$$

It can also be shown after some algebra that

$$H(0) = H(1) = 4\pi; \qquad H'(1) = -H'(0) = \pi. \tag{36}$$

Equation (33) can be rewritten as

$$\widetilde{\Phi}^s_{\ell m}(\alpha) = \frac{C_{\ell 0} H(\alpha) - 2}{D_\ell(\alpha)} \widetilde{\Phi}^e_{\ell m}(\alpha), \tag{37}$$

where

$$D_\ell(\alpha) \equiv \alpha(\alpha-1) + [2 - \ell(\ell+1)] - C_{\ell 0} H(\alpha). \tag{38}$$

Both $H(\alpha)$ and $D_\ell(\alpha)$ are even functions of $\alpha - \frac{1}{2}$.

Thus the linear response of the singular isothermal sphere to adiabatic perturbations for odd ℓ is analytic and for even ℓ requires only the numerical evaluation of a single function $H(\alpha)$.

3.1. Zero-frequency normal modes

Roots of the dispersion relation $D_\ell(\alpha) = 0$ may be regarded as zero-frequency normal modes of the singular isothermal sphere. The contribution to $\Phi^s_{\ell m}(r)$ from each root of $D_\ell(\alpha)$ is proportional to $r^{-\alpha}$, which follows from the inverse Mellin transform. In general these solutions should not be interpreted as physical normal modes, since a power-law potential perturbation is always nonlinear at large or small radii, except in the special case $Re(\alpha) = 0$. In the fluid case, this limitation is highlighted by Lichtenstein's theorem, which states that static, isolated, equilibrium fluid systems *must* be spherical, i.e. there are no finite normal modes with zero frequency (e.g. Lindblom 1992).

Two trivial normal modes arise from gauge transformations: the mode $\alpha = 0$, $\ell = 0$ corresponds to a shift in the zero-point of the potential, and $\alpha = 1$, $\ell = 1$ corresponds to a uniform translation of the unperturbed stellar system.

The symmetries (35) imply that if α is a root of the dispersion relation, then so is α^* and $1-\alpha$. Thus if $x \equiv \alpha - \frac{1}{2}$ the roots come in pairs $\frac{1}{2}+x$, $\frac{1}{2}-x$ when x is real or imaginary, and in quartets $\frac{1}{2}+x$, $\frac{1}{2}-x$, $\frac{1}{2}+x^*$, $\frac{1}{2}-x^*$ if x is complex.

Finding the roots of $D_\ell(\alpha)$ is easy when ℓ is odd: then $C_{\ell 0} = 0$ so that the roots are

$$\alpha = \tfrac{1}{2} \pm \left[\ell(\ell+1) - \tfrac{7}{4}\right]^{1/2}, \qquad \ell \text{ odd.} \tag{39}$$

To find the roots when ℓ is even, it is convenient to rewrite the function $H(\alpha)$ explicitly as a complex function of a complex variable. Defining $\alpha = a + bi$ where a and b are real, we find the real and imaginary parts of $H(\alpha)$ to be

$$\begin{aligned} Re[H(\alpha)] &= \frac{2^{5/2}}{\pi^{1/2}e}\int \frac{RdR}{g(R)}\int dw_1 \int dw_1' \cos[b\ln(x/x')]x^{a-1}x'^{-a}, \\ Im[H(\alpha)] &= \frac{2^{5/2}}{\pi^{1/2}e}\int \frac{RdR}{g(R)}\int dw_1 \int dw_1' \sin[b\ln(x/x')]x^{a-1}x'^{-a}. \end{aligned} \tag{40}$$

The roots of $D_\ell(\alpha)$ occur at the simultaneous zeros of its real and imaginary parts. Figure 1 shows the zero curves in the complex plane of the real and imaginary parts of $D_\ell(\alpha)$, for several values of ℓ. The roots with the smallest values of $|a| = |\,Re(\alpha)|$, which we denote $\alpha = a_0^+ \pm ib_0$ and $a_0^- \pm ib_0$, have the greatest physical significance, since these generally determine the asymptotic behaviour of the response of the system at small and large radii. These roots are given in Table 1; note that they satisfy the constraint $-\ell \le a_-, a_+ \le \ell+1$ given by equation (20). Also note that $a_0^- + a_0^+ = 1$.

The normal mode corresponding to the root $a_0^+ = 1$ for $\ell = 1$ is simply a uniform displacement of the unperturbed system.

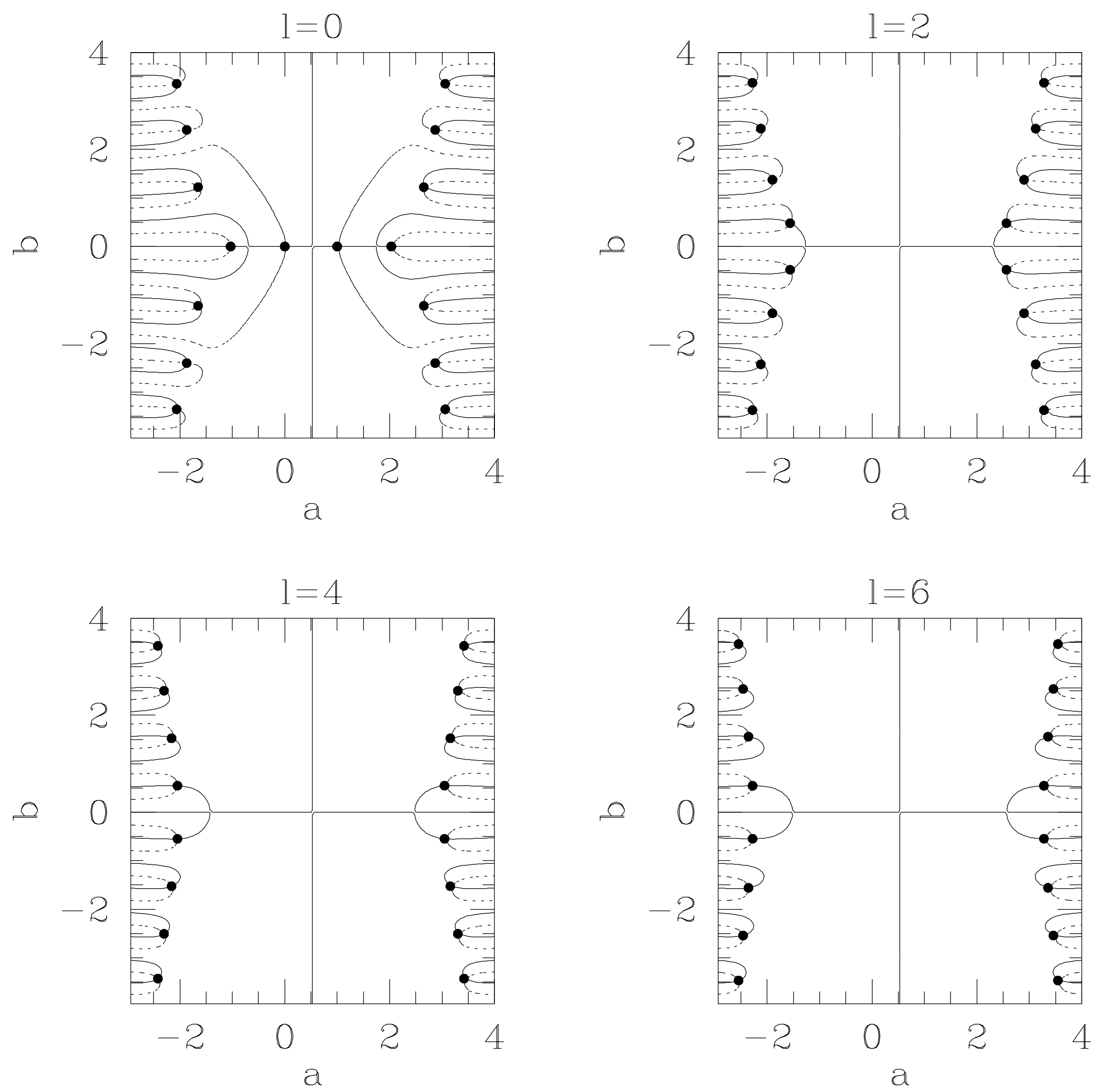

Fig. 1.— The zero curves of the real (dotted) and imaginary (solid) parts of $D_\ell(\alpha)$ (eq. 38) for indicated values of ℓ. The roots (solid points) lie at the intersections of the two curves. Those with largest real parts in the left half-plane will dominate the response at small radii.

For comparison, the dispersion relation in the fluid system is obtained by setting $H(\alpha) = 0$ in equation (38). The corresponding roots are given by (39) for all ℓ, both even and odd. Thus for $\ell = 0$ the fluid system has $\alpha = \frac{1}{2}(1 \pm i\sqrt{7})$; this is a familiar result since the difference in potential between the singular and non-singular isothermal spheres oscillates with this spatial frequency at large radii (Chandrasekhar 1939). For $\ell = 2$, equation (39) yields $\alpha = \frac{1}{2}(1 \pm \sqrt{17})$.

3.2. Green's function

We now discuss the adiabatic response of the singular isothermal sphere to a point mass m_p at position $\mathbf{r}_p$, which corresponds to the external potential

$$\Phi^e(\mathbf{r}) = \sum_{\ell m} \Phi^e_{\ell m}(r) Y_{\ell m}(\mathbf{\Omega}), \qquad \text{where} \qquad \Phi^e_{\ell m}(r) = -\frac{4\pi G m_p}{2\ell+1} Y^*_{\ell m}(\mathbf{\Omega}_p) \frac{r_<^\ell}{r_>^{\ell+1}}. \tag{41}$$

Here $\mathbf{\Omega}_p$ denotes the angular coordinates of $\mathbf{r}_p$, $r_< = \min(|\mathbf{r}|, |\mathbf{r}_p|)$ and $r_> = \max(|\mathbf{r}|, |\mathbf{r}_p|)$. The Mellin transform of the point-mass potential is therefore given by

$$\widetilde{\Phi}^e_{\ell m}(\alpha) = -\frac{4\pi G m_p}{2\ell+1} Y^*_{\ell m}(\mathbf{\Omega}_p) r_p^{\alpha-1} \left(\frac{1}{\alpha+\ell} - \frac{1}{\alpha-\ell-1} \right), \qquad -\ell < Re(\alpha) < \ell + 1. \tag{42}$$

We now employ equation (37) to find the response

$$\widetilde{\Phi}^s_{\ell m}(\alpha) = -\frac{4\pi G m_p}{2\ell+1} Y^*_{\ell m}(\mathbf{\Omega}_p) r_p^{\alpha-1} \left(\frac{1}{\alpha+\ell} - \frac{1}{\alpha-\ell-1} \right) \frac{C_{\ell 0} H(\alpha) - 2}{D_\ell(\alpha)}. \tag{43}$$

Table 1: Asymptotically dominant roots

ℓ	a_0^-	a_0^+	b_0
0	0.0000	1.0000	0.0000
1	0.0000	1.0000	0.0000
2	−1.5623	2.5623	0.4804
3	−2.7016	3.7016	0.0000
4	−2.0460	3.0460	0.5480
5	−4.8151	5.8151	0.0000
6	−2.2771	3.2771	0.5457
7	−6.8655	7.8655	0.0000

The inverse Mellin transform is

$$\begin{aligned}\Phi^s_{\ell m}(r) &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\widetilde{\Phi}^s_{\ell m}(\alpha)r^{-\alpha}d\alpha \\ &= \frac{Gm_p}{r_p}\frac{2i}{2\ell+1}Y^*_{\ell m}(\mathbf{\Omega}_p)\int_{a-i\infty}^{a+i\infty}d\alpha\left(\frac{r_p}{r}\right)^\alpha\left(\frac{1}{\alpha+\ell}-\frac{1}{\alpha-\ell-1}\right)\frac{C_{\ell 0}H(\alpha)-2}{D_\ell(\alpha)}.\end{aligned} \tag{44}$$

The constant a must be chosen so that $-\ell < a < \ell+1$ and $a_- < a < a_+$ (eq. 20). If we choose $a = \frac{1}{2}$ and use the fact that $H(\alpha)$ and $D_\ell(\alpha)$ are even functions of $\alpha - \frac{1}{2}$, then it is easy to show that

$$\Phi^s_{\ell m}(r) = \frac{1}{(rr_p)^{1/2}}\Psi_{\ell m}\left(\frac{r}{r_p}\right), \qquad \text{where} \qquad \Psi_{\ell m}(u) = \Psi_{\ell m}(1/u); \tag{45}$$

in other words the response for $r < r_p$ is determined by the response for $r > r_p$.

To evaluate the integral (44) in closed form, we analytically continue the integrand over the whole complex plane and then close the integration contour, in the left half-plane for $r \le r_p$ and in the right half-plane for $r > r_p$. The contributions to the integral come from the poles at $\alpha = -\ell$, $\alpha = \ell+1$, and the roots of $D_\ell(\alpha)$, which we denote by $\alpha = \frac{1}{2} \pm x_k$, where $Re(x_k) > 0$. The asymptotic behavior at small radii, $\Phi^s_{\ell m} \sim r^{-a_-}$, is determined by the pole in the negative half-plane with the smallest $|\,Re(\alpha)|$ (either a_0^- from Table 1 or $-\ell$), while the asymptotic behavior at large radii, $\Phi^s_{\ell m} \sim r^{-a_+}$, is determined by the pole in the positive half-plane with the smallest $|\,Re(\alpha)|$ (either a_0^+ from Table 1 or $\ell+1$). Thus

$$\begin{aligned}\Phi^s_{\ell m}(r) &= \frac{4\pi Gm_p}{(2\ell+1)r_p}Y^*_{\ell m}(\mathbf{\Omega}_p)\left[\left(\frac{r}{r_p}\right)^\ell + \sum_k\frac{2\ell+1}{D'_\ell(\frac{1}{2}-x_k)}\left(\frac{r}{r_p}\right)^{x_k-\frac{1}{2}}\right], \quad r \le r_p, \\ &= \frac{4\pi Gm_p}{(2\ell+1)r_p}Y^*_{\ell m}(\mathbf{\Omega}_p)\left[\left(\frac{r_p}{r}\right)^{\ell+1} - \sum_k\frac{2\ell+1}{D'_\ell(\frac{1}{2}+x_k)}\left(\frac{r_p}{r}\right)^{x_k+\frac{1}{2}}\right], \quad r > r_p;\end{aligned} \tag{46}$$

these two expressions satisfy the symmetry relation (45). The sum is over all roots with $Re(\alpha - \frac{1}{2}) > 0$; recall that if $Im(\alpha) \ne 0$ then α^* is also a root.

The first term on the right side of each equality defines a response that precisely cancels the external perturbing potential $\Phi^e_{\ell m}$ (eq. 41). The total potential perturbation $\Phi^t_{\ell m} = \Phi^e_{\ell m} + \Phi^s_{\ell m}$ therefore is entirely determined by the spectrum of normal modes. When ℓ is even the roots x_k are generally complex (Table 1) so $\Phi^t_{\ell m}(r)$ varies sinusoidally in $\log r$; for odd ℓ the total potential has no phase variation because the roots are real.

Since $D'_\ell(\alpha)$ is an odd function of $\alpha - \frac{1}{2}$, the response potential $\Phi^s_{\ell m}(r)$ is continuous through the shell $r = r_p$. The total response $\Phi_s(\mathbf{r}) = \sum_{\ell,m}\Phi^s_{\ell m}(r)Y_{\ell m}(\mathbf{\Omega})$ is also continuous.

In addition, $d\Phi^s_{\ell m}/dr$ is continuous through the shell $r = r_p$; this can be shown using the identity

$$\sum_k \frac{x_k}{D'_\ell(\frac{1}{2}+x_k)} = \tfrac{1}{2}, \tag{47}$$

which in turn can be derived by considering the integral $\int_C d\alpha(\alpha - \frac{1}{2})/D_\ell(\alpha)$ where C is the circle $|\alpha| \to \infty$.

In the case of a fluid system, the analogous expressions are

$$\begin{aligned}
\Phi^s_{\ell m}(r) &= \frac{4\pi G m_p}{(2\ell+1)r_p} Y^*_{\ell m}(\mathbf{\Omega}_p)\left[\left(\frac{r}{r_p}\right)^{\ell} - \frac{2\ell+1}{[4\ell(\ell+1)-7]^{1/2}}\left(\frac{r}{r_p}\right)^{[\ell(\ell+1)-7/4]^{1/2}-1/2}\right], \quad r \le r_p \\
&= \frac{4\pi G m_p}{(2\ell+1)r_p} Y^*_{\ell m}(\mathbf{\Omega}_p)\left[\left(\frac{r_p}{r}\right)^{\ell+1} - \frac{2\ell+1}{[4\ell(\ell+1)-7]^{1/2}}\left(\frac{r_p}{r}\right)^{[\ell(\ell+1)-7/4]^{1/2}+1/2}\right], \quad r > r_p.
\end{aligned} \tag{48}$$

A similar result was derived already by Lynden-Bell (1985), with minor differences—in particular he assumed that the fluid halo stopped at r_p, which changes $\Phi^t_{\ell m}$ by a multiplicative factor.

Some care is required to interpret these expressions when $\ell = 0$ or $\ell = 1$. For $\ell = 0$, the response potential interior to the point mass contains constant terms ($\propto r^\ell$ and $\propto r^{x_k-1/2}$ for $x_k = \frac{1}{2}$); these terms exert no force and can be eliminated by re-defining the zero-point of the potential. When $\ell = 1$, the concept of the linear adiabatic response to a small perturbing potential is ill-defined, since a small but steady perturbing potential can lead to a large shift in the center of mass of the stellar system. This indeterminacy can be reflected in equations (46) and (48) by adding an arbitrary amount of the normal mode $\Phi^s_{1m} \propto r^{-1}$ that corresponds to a uniform translation of the unperturbed stellar system.

In practice, it is easiest to determine the response potential through direct numerical calculation of the inverse Mellin transform (44), rather than by summing the residues at all the poles. The simplest choice of contour is along the symmetry axis, $a = \frac{1}{2}$, for which the imaginary part of $H(\alpha)$ vanishes. The real part of equation (40) then becomes

$$H(\tfrac{1}{2}+ib) = \frac{2^{5/2}}{\pi^{1/2}e}\int \frac{RdR}{g(R)}\left\{\left[\int dw_1 \cos(b\ln x)x^{-1/2}\right]^2 + \left[\int dw_1 \sin(b\ln x)x^{-1/2}\right]^2\right\}, \tag{49}$$

which has a narrow peak about $b = 0$ and decays approximately as b^{-1} for $b \gtrsim 50$.

It is also helpful to rewrite equation (44) as a Fourier transform which can be evaluated

using FFTs (e.g. Acton 1990):

$$\Phi^s_{\ell m}(r) = -\frac{4Gm_p}{(r_p r)^{1/2}} Y^*_{\ell m}(\mathbf{\Omega}_p) \int_0^\infty db \cos\left[b \ln(r/r_p)\right] \frac{C_{\ell 0} H(\frac{1}{2}+ib) - 2}{[(\frac{1}{2}+\ell)^2 + b^2] D_\ell(\frac{1}{2}+ib)}, \tag{50}$$

The integration parameter b represents the logarithmic wavenumber of the response. In the fluid case,

$$\Phi^s_{\ell m}(r) = -\frac{4Gm_p}{(r_p r)^{1/2}} Y^*_{\ell m}(\mathbf{\Omega}_p) \int_0^\infty db \cos\left[b \ln(r/r_p)\right] \frac{2}{[(\frac{1}{2}+\ell)^2 + b^2][(\frac{1}{2}+\ell)^2 + b^2 - 2]}. \tag{51}$$

3.3. Density response

The density response is determined directly from the Green's function (46) through Poisson's equation. Write the density as a multipole expansion:

$$\rho^s(\mathbf{r}) = \sum_{\ell m} \rho^s_{\ell m}(r) Y_{\ell m}(\mathbf{\Omega}); \tag{52}$$

then substitute into Poisson's equation so that

$$4\pi G \rho^s_{\ell m}(r) = \frac{1}{r^2}\frac{d}{dr} r^2 \frac{d\Phi^s_{\ell m}}{dr} - \frac{\ell(\ell+1)}{r^2}\Phi^s_{\ell m}. \tag{53}$$

There is no surface-density layer at $r = r_p$, because the response potential and its gradient are continuous at $r = r_p$ (see the discussion following equation 46).

Substituting equation (46) for $\Phi^s_{\ell m}$ gives

$$\begin{aligned} \rho^s_{\ell m}(r) &= \frac{m_p}{r_p^3} Y^*_{\ell m}(\mathbf{\Omega}_p) \sum_k \left(\frac{r}{r_p}\right)^{x_k - 5/2} \frac{[x_k^2 - (\ell+\frac{1}{2})^2]}{D'_\ell(\frac{1}{2} - x_k)}, \qquad r \le r_p \\ &= \frac{m_p}{r_p^3} Y^*_{\ell m}(\mathbf{\Omega}_p) \sum_k \left(\frac{r_p}{r}\right)^{x_k + 5/2} \frac{[x_k^2 - (\ell+\frac{1}{2})^2]}{D'_\ell(\frac{1}{2} - x_k)}, \qquad r > r_p. \end{aligned} \tag{54}$$

At small radii, the fractional density perturbation is $\rho^s_{\ell m}/\rho_0 \sim r^{-a_0^-}$, while at large radii the fractional perturbation varies as $r^{-a_0^+}$ (cf. Table 1).

For a fluid system,

$$\begin{aligned} \rho^s_{\ell m}(r) &= \frac{m_p}{r_p^3} \frac{2Y^*_{\ell m}(\mathbf{\Omega}_p)}{[4\ell(\ell+1) - 7]^{1/2}} \left(\frac{r}{r_p}\right)^{[\ell(\ell+1)-7/4]^{1/2} - 5/2}, \quad r \le r_p \\ &= \frac{m_p}{r_p^3} \frac{2Y^*_{\ell m}(\mathbf{\Omega}_p)}{[4\ell(\ell+1) - 7]^{1/2}} \left(\frac{r_p}{r}\right)^{[\ell(\ell+1)-7/4]^{1/2} + 5/2}, \quad r > r_p. \end{aligned} \tag{55}$$

3.4. Amplification factor

It is useful to express these results in terms of the overall amplification of the point-mass perturbation by the response. Defining the amplification $\chi_\ell(r) = (\Phi^s_{\ell m} + \Phi^e_{\ell m})/\Phi^e_{\ell m}$, we find that

$$\begin{aligned} \chi_\ell(r) &= -\sum_k \frac{2\ell+1}{D'_\ell(\frac{1}{2}-x_k)}\left(\frac{r}{r_p}\right)^{x_k-\ell-\frac{1}{2}}, \qquad r \le r_p \\ &= \sum_k \frac{2\ell+1}{D'_\ell(\frac{1}{2}+x_k)}\left(\frac{r_p}{r}\right)^{x_k-\ell-\frac{1}{2}}, \qquad r > r_p. \end{aligned} \tag{56}$$

The amplification also obeys a symmetry relation similar to (45), $\chi_\ell(r) = \psi_\ell(r/r_p)$ where $\psi_\ell(u) = \psi_\ell(1/u)$.

As we discussed after equation (48) these formulae are not meaningful for $\ell = 0$ when $r < r_p$, or for $\ell = 1$.

Asymptotically, $\chi_\ell(r) \sim r^{x_0-\ell-1/2} = r^{-a_0^- - \ell}$ as $r \to 0$, and $\chi_\ell(r) \sim r^{-x_0+\ell+1/2} = r^{-a_0^+ + \ell + 1}$ as $r \to \infty$. Values of a_0 are given in Table 1 and we see that for all $\ell > 0$ the amplification diverges as $r \to 0$ or $r \to \infty$. However, for even ℓ this asymptotic behavior does not necessarily appear until very large values of $|\log r|$ are reached. For odd ℓ, there is no distinction between the exact and asymptotic behavior since the response is determined by a single root.

4. Applications

4.1. Tidal amplification

Consider the response of the halo to a point-mass satellite on a circular orbit of radius r_p. Our assumption that the satellite perturbation is adiabatic is not accurate at radii $r \sim r_p$, where the characteristic orbital period in the halo is comparable to the satellite's orbital period. Nevertheless, the adiabatic approximation is plausible at radii $r \ll r_p$, and should approximately describe how the tidal field of the satellite is propagated to small radii. The monopole ($\ell = 0$) response of the halo is not so interesting, since it is difficult to distinguish observationally from the potential of the unperturbed stellar system. As we have discussed, our formalism is not powerful enough to determine the dipole ($\ell = 1$) response. Thus we shall focus on the quadrupole ($\ell = 2$) response.

Figure 2 shows the amplification $\chi_2(r)$ of the quadrupole tidal field from a point mass at radius r_p. The amplification has a weak maximum of about 2 at $\log(r/r_p) \approx -1$ and becomes

negative for $\log r/r_p \lesssim -2$. This oscillation arises because the asymptotically dominant root in Table 1 is complex. The amplification remains negative down to $\log r/r_p \approx 5$—far too small to be of interest—where it begins a positive rise. The dashed line shows the much stronger quadrupole amplification for the analogous fluid system.

These results indicate that the response of an isothermal halo does not greatly enhance the quadrupole tidal field from an external point mass: the response is much smaller than for an isothermal fluid and over a large range in $\log(r/r_p)$ the halo response actually screens or suppresses the external tidal field.

4.2. Response to disk growth

We can also calculate the response of the halo to the adiabatic growth of an embedded disk (Binney & May 1986; Dubinski 1994). We compute the response using scale-free, axisymmetric, razor-thin disk models, which have density distributions

$$\rho(\mathbf{r}) = \Sigma(r)\frac{\delta(\cos\theta)}{r}. \tag{57}$$

We shall use the relation

$$\delta(\cos\theta) = 2\pi \sum_{\ell} Y^{*}_{2\ell,0}(\tfrac{1}{2}\pi, 0) Y_{2\ell,0}(\theta, 0); \tag{58}$$

note that only even harmonics contribute to the sum.

Using equation (46), we calculate the total potential arising from the embedded disk:

$$\begin{aligned} \Phi^{t}_{2\ell,0}(r) &= 8\pi^2 G Y^{*}_{2\ell,0}\left(\tfrac{1}{2}\pi, 0\right) \sum_k \left\{ \frac{1}{D'_{2\ell}(\frac{1}{2} - x_k)} \int_r^\infty dr' \left(\frac{r}{r'}\right)^{x_k - \frac{1}{2}} \Sigma(r') \right. \\ &\quad \left. - \frac{1}{D'_{2\ell}(\frac{1}{2} + x_k)} \int_0^r dr' \left(\frac{r'}{r}\right)^{x_k + \frac{1}{2}} \Sigma(r') \right\}. \end{aligned} \tag{59}$$

For example, in the case of a Mestel disk, $\Sigma(r) = \Sigma_a r_a / r$, the monopole potential is

$$\Phi^{t}_{00}(r) = \frac{4\pi^{3/2} G \Sigma_a r_a}{D'_0(1)} \ln r; \tag{60}$$

here we have neglected constant contributions to the potential (including a divergent one) which change the zero-point but have no physical effects. For $\ell > 0$, we have

$$\Phi^{t}_{2\ell,0}(r) = 16\pi^2 G \Sigma_a r_a Y^{*}_{2\ell,0}\left(\tfrac{1}{2}\pi, 0\right) \sum_k \frac{x_k}{D'_{2\ell}(\frac{1}{2} + x_k)(\frac{1}{4} - x_k^2)}. \tag{61}$$

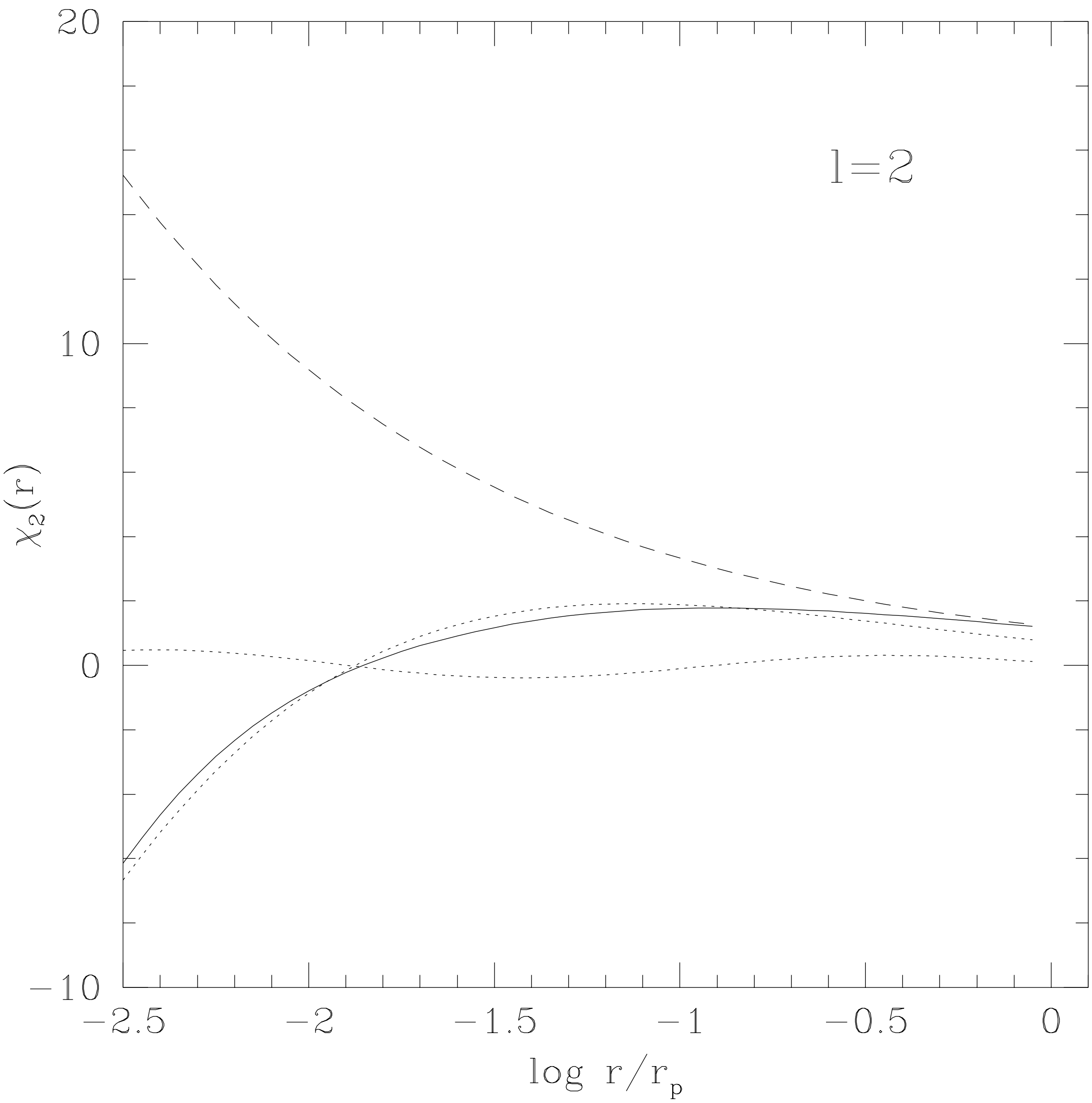

Fig. 2.— Amplification $\chi_2(r)$ of the quadrupole tidal field from a point mass at r_p. The solid line shows the exact result from numerical evaluation of equation (56). The dotted line that follows the solid line shows the amplification due to the asymptotically dominant root. The dotted line that oscillates weakly about zero shows the contribution from the next most important root. The dashed line shows the much stronger amplification for the fluid case.

These terms give rise to tangential forces but no radial forces.

We can evaluate sums over roots by analogy with equation (47), using the integral $\int_C d\alpha(\alpha - \frac{1}{2})/D_\ell(\alpha)(\alpha + 1 - \gamma)(\alpha - 2 + \gamma)$. This leads to the identity

$$\sum_k \frac{x_k}{D'_\ell(\frac{1}{2} - x_k)[(\frac{3}{2} - \gamma)^2 - x_k^2]} = \frac{1}{2D_\ell(2 - \gamma)}. \tag{62}$$

Using this result for $\gamma = 1$ and the identity (36), equation (61) simplifies to

$$\Phi^t_{2\ell,0}(r) = \frac{8\pi^2 G\Sigma_a r_a Y^*_{2\ell,0}(\pi/2, 0)}{2 - 2\ell(2\ell + 1) - 4\pi C_{2\ell,0}}, \tag{63}$$

which, remarkably, is completely analytic.

We can easily extend these results to disks with surface density $\Sigma(r) = \Sigma_a(r_a/r)^\gamma$, $1 < \gamma < 2$. The total potential is

$$\Phi^t_{2\ell,0}(r) = \frac{8\pi^2 G\Sigma_a r_a}{D_{2\ell}(2 - \gamma)} \left(\frac{r_a}{r}\right)^{\gamma - 1} Y^*_{2\ell,0}\left(\tfrac{1}{2}\pi, 0\right). \tag{64}$$

We can also express this result in terms of the amplification $\chi_{2\ell} = \Phi^t_{2\ell,0}/\Phi^d_{2\ell,0}$, where Φ^d is the direct potential from the disk. We have

$$\chi_{2\ell} = \frac{(1 - \gamma)(2 - \gamma) - 2\ell(2\ell + 1)}{(1 - \gamma)(2 - \gamma) - 2\ell(2\ell + 1) + 2 - C_{2\ell}H(2 - \gamma)}. \tag{65}$$

This expression is valid for all values of 2ℓ and $\gamma \in [1, 2]$ except for $2\ell = 0, \gamma = 1$. Note that for large ℓ, $\chi_{2\ell} \to 1$; the halo does not amplify the response from the disk on small scales. Table 2 shows values of $\chi_{2\ell,0}$ for a range of 2ℓ and γ.

Table 2: Disk amplification

$2\ell/\gamma$	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$
0	2.00	1.93	1.90	1.93
2	1.39	1.33	1.33	1.33
4	1.10	1.10	1.10	1.10
6	1.05	1.05	1.05	1.05
8	1.03	1.03	1.03	1.03

Using Poisson's equation, we derive the response density of the halo in the presence of the disk. The response consists of a density

$$\rho^s = \frac{2\pi\Sigma_a r_a^\gamma}{r^{\gamma+1}} \sum_\ell (\chi_{2\ell} - 1) Y^*_{2\ell,0}(\tfrac{1}{2}\pi, 0) Y_{2\ell,0}(\theta, 0); \tag{66}$$

This can be expressed as an induced enhancement δ in the local halo density

$$\delta \equiv \frac{\rho^s}{\rho_0} = \frac{2\pi\Sigma_a r_a^{\gamma-2}}{\rho_a r^{\gamma-1}} \sum_\ell (\chi_{2\ell}-1) Y^*_{2\ell,0}(\tfrac{1}{2}\pi,0) Y_{2\ell,0}(\theta,0), \tag{67}$$

where ρ_a is defined in equation (16). Figure 3 shows contours of the fractional density enhancement δ for various values of γ.

4.3. Response to black hole growth

We may also consider the response of the halo to the adiabatic growth of a central dark object, such as a massive black hole. In this case the nonlinear problem has already been investigated and solved numerically by several authors (Peebles 1972; Young 1980; Quinlan et al. 1995), so the results of our linear calculation are only of academic interest.

We take the limit of the second of equations (46) in which $r_p \to 0$, and consider the monopole case $\ell = m = 0$. Only the term corresponding to the dominant root $\frac{1}{2} + x_k = 1$ survives since $\Phi^s_{\ell m} \propto r_p^{x_k - 1/2}$ and all other terms have $Re(x_k) > \frac{1}{2}$. The total potential is therefore

$$\Phi^t(r) = (\Phi^e_{00} + \Phi^s_{00})\, Y_{00}(\mathbf{\Omega}) = -\frac{Gm_p}{r}\left[D_0'(1)\right]^{-1}, \tag{68}$$

where $D_0'(1) = \frac{1}{2}$ so that the linear response amplifies the direct black-hole potential by a factor of 2.

The interpretation of this result is as follows. At radii $r \lesssim r_h \equiv \beta G m_p$ the black hole induces a (nonlinear) density cusp in the stellar system. At radii $r \gg r_h$ we might expect a linear density response, but, as it turns out, the linear terms vanish so the response is second-order in the black-hole mass (this is straightforward to show in the case of a system composed of stars on circular orbits). Thus the total potential perturbation at large radii consists of the Keplerian potential of the black hole augmented by the mass in the density cusp. The particular value of the augmented mass given above, $m_p/D_0'(1)$, is not necessarily accurate since it is derived from a linear calculation while the cusp is nonlinear.

5. Discussion

We have discussed the response of a self-similar stellar system to weak adiabatic gravitational perturbations. Our problem is idealized because real stellar systems are only approximately self-similar, and because we neglect the time-dependence that accompanies most tidal fields, such as those from orbiting satellites. Nevertheless, this calculation provides one of the very

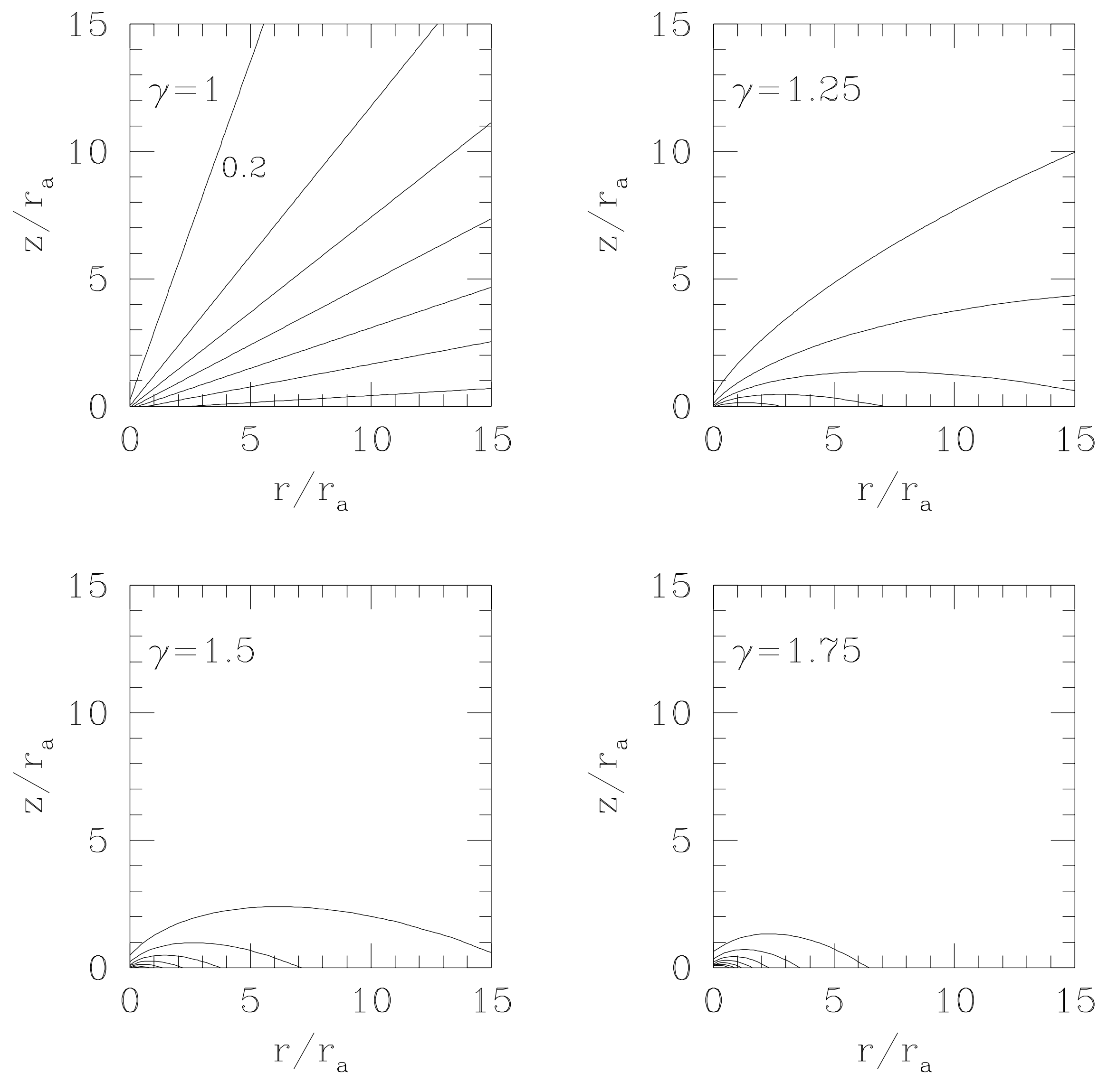

Fig. 3.— Contours of the fractional halo response density δ (eq. 67) induced by razor-thin disks with surface density $\Sigma_a(r_a/r)^\gamma$. Contours show levels $\delta = 0.2, 0.3, \ldots, 0.8$ in units of $\Sigma_a/r_a\rho_a$, starting from the topmost contour in each panel and increasing toward the disk.

few examples where the linear response of an inhomogeneous stellar system can be explicitly computed in terms of quadratures, and hence offers analytic insight into the nature of the response. Our results can also be used to test numerical codes used in linear response calculations.

Our assumptions that the stellar system is self-similar and that the perturbation is adiabatic can be lifted in more realistic calculations using matrix methods, such as those of Weinberg (1995, 1997). Weinberg shows that in some cases individual resonances in the stellar system can lead to strong enhancements in the external tidal field over a limited range of radii.

Using the methods derived above, we have investigated the propagation of the tidal disturbance from a static satellite into the inner galaxy; in effect, we have computed the static Love number for an isothermal stellar system. For odd spherical harmonics (other than $\ell = 1$, for which the adiabatic approximation is not self-consistent), the response of the stellar system is identical to the response of the barotropic fluid system with the same density profile. For even $\ell > 0$, we find that the zero-frequency normal modes of the isothermal sphere, which have the spatial dependence $\exp(-\alpha \log r) Y_{\ell m}(\mathbf{\Omega})$, have complex spatial eigenvalues α and hence oscillate in $\log r$ at fixed angular position $\mathbf{\Omega}$. Partly because of this oscillatory behavior, the halo response does not strongly amplify the satellite's tidal field in the inner galaxy: the amplification factor or Love number for the dominant quadrupole ($\ell = 2$) tidal component is never more than about 2 for $0.01 \lesssim r/r_p \lesssim 1$, crosses zero near $r/r_p \simeq 0.01$ and is negative over the next decade in radius (Fig. 2).

Our calculations have some similarity to Young's (1980) investigation of the response of an isothermal stellar system to the adiabatic growth of a central black hole. The principal differences are that (i) Young's method is fully nonlinear, whereas ours is only linear; (ii) our method can be applied to non-spherical perturbations, such as the slow growth of a disk.

Finally, although we have examined only the singular isothermal stellar system with density $\rho_0(r) \propto r^{-2}$, it is straightforward to extend these calculations to other self-similar systems with density $\rho_0(r) \propto r^{-k}$.

This research was supported by NSERC, the Fund for Astrophysical Research (C.M.) and an Imasco Fellowship (S.T.).

REFERENCES

Acton, F. S. 1990, Numerical Methods That Work (Washington D.C.: Mathematical Association of America)

Binney, J. J. & May, A. 1986, MNRAS 218, 743

Binney, J. J., & Tremaine, S. 1987, Galactic Dynamics (Princeton: Princeton University Press)

Chandrasekhar, S. 1939, An Introduction to the Study of Stellar Structure (Chicago: University of Chicago Press).

Dubinski, J. 1994, ApJ 431, 617

Edmonds, A. R. 1960, Angular Momentum in Quantum Mechanics (Princeton: Princeton University Press).

Jackson, J. D. 1975, Classical Electrodynamics (New York: John Wiley & Sons)

Jeffreys, H. 1970, The Earth, 5th Ed. (Cambridge: Cambridge University Press)

Kalnajs, A. J. 1971, ApJ 166, 275

Lindblom, L. 1992, Phil. Trans. Roy. Soc. A, 340, 353

Lynden-Bell, D. 1969, MNRAS 144, 189

Lynden-Bell, D. 1985, in The Milky Way Galaxy, eds. H. van Woerden et al. (Dordrecht: Reidel), 461

Nelson, R. W. & Tremaine, S. 1995, in Gravitational Dynamics, eds. O. Lahav, E. Terlevich and R. Terlevich (Cambridge: Cambridge University Press), 73

Peebles, P.J.E. 1972, Gen. Rel. Grav. 3, 63

Quinlan, G. D., Hernquist, L., & Sigurdsson, S. 1995, ApJ 440, 554

Tremaine, S. & Weinberg, M. D. 1984, MNRAS 209, 729

Weinberg, M. D. 1989, MNRAS 239, 549

Weinberg, M. D. 1995, ApJL, 455, L31

Weinberg, M. D. 1997, Dynamics of an interacting luminous disk, dark halo, and satellite companion, astro-ph/9707189. Submitted to MNRAS

Young, P. 1980, ApJ 242, 1232